

Stability of combined natural and forced cross flow in a vertical slot

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Abstract—This study deals with the stability of combined natural and forced cross flow of fluids contained in a vertical slender slot with thin porous walls. The system parameters involved wide ranges of Prandtl and Peclet numbers, viz. $0.72 \leq Pr \leq 100$ and $0 \leq Pe \leq 20$ respectively. In addition, this problem is investigated for two different thermal boundary conditions at the hot wall, namely prescribed temperature and heat flux. When the results of this investigation were compared with the natural convection problem, it was found that for $Pr > 0.72$ and $Pe < 4$ the cross flow has a destabilizing effect, while for $Pe > 4$ the flow is stabilized. Furthermore, increasing the cross flow shifts the mode of instability, for fluids having small Prandtl numbers, from stationary to travelling waves.

1. INTRODUCTION

THE STABILITY of natural convection of fluids contained in a vertical slot heated from one side has received the attention of many investigators due to its wide applications. A few investigators were concerned with the conduction regime [1, 2], while others studied the boundary-layer regime [3–5]. The stability of conduction regime was first investigated by Gerstuni [1] who considered fluids of small Prandtl numbers, and obtained highly approximate curves of neutral stability for the case of stationary disturbances.

In general, for small Prandtl number, Pr , it was concluded that the natural convective flow in a vertical slot is unstable with respect to certain stationary modes of disturbances which had been analytically proved and experimentally observed. Rudakov [2] studied this problem for $0.01 \leq Pr \leq 10$ and found that the instability, for the conduction regime, sets in as two-dimensional horizontal cells at Grashof number of about 7700. In addition, this critical Grashof number, Gr_c , was suggested to be a weak function of Pr . Furthermore, Vest and Arpaci [6] confirmed these results. On the other hand, for large Pr Gill and Kirkham [7] found that the instability sets in as travelling waves.

Korpela *et al.* [8] studied the effect of Pr on the onset of instability for the conduction regime in a vertical slender slot at different prescribed wall temperatures. Their results indicated that for $Pr < 12.7$ the instability sets in as stationary horizontal cells, with Gr_c nearly independent of Pr . Conversely, for $Pr > 12.7$, the instability sets in as travelling transverse waves with Gr_c decreasing monotonically with Pr .

The effect of convective boundary conditions, was studied by Özişik and Hassab [9]. Their results reveal that the heat transfer coefficient expressed by the Biot number, has a significant destabilizing effect on the flow field only under the condition where the instability sets in as travelling waves. In addition the value of Pr , which is associated with transition in the instability from

stationary to travelling waves monotonically decreases with Biot number.

To the authors' knowledge there are no publications dealing with the stability of combined cross flow and natural convection in a vertical slot. Therefore, this investigation is devoted to the study of the influence of cross flow on the stability of natural convection in a vertical slender slot with porous walls.

2. ANALYSIS

The problem geometry is schematically illustrated in Fig. 1. When a fluid is contained in a vertical slender slot ($L/d \gg 1$) which is differentially heated, an initial unicellular motion of the fluid is developed in the slot. If a uniform forced cross flow, of velocity U_0 , is applied between the two vertical porous walls the problem becomes more involved. In formulating this problem, the fluid is considered Newtonian, incompressible, and the Boussinesq approximation is applied. In addition, while the cold wall is kept at constant temperature, the hot wall is assumed either with constant temperature or constant heat flux.

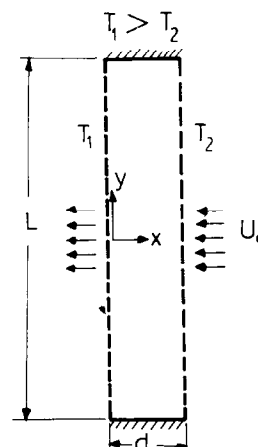


FIG. 1. Problem geometry.

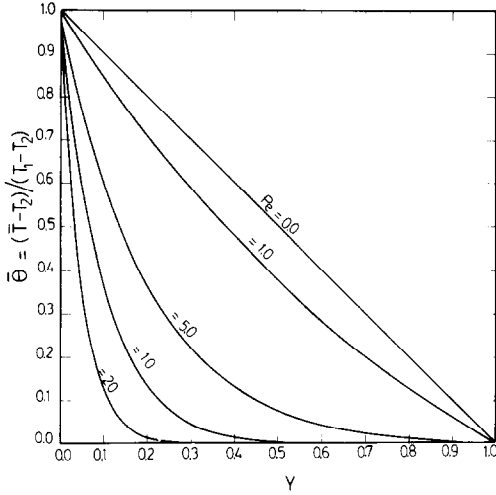


FIG. 3. Effect of cross flow on the base flow temperature profile.

can be easily seen that as Pe increases the velocity profile is damped and shifted towards the hot wall. In addition, the cross flow increases the rate of heat transfer from the hot wall to the fluid. Conversely, the heat transfer from the fluid to the cold wall is decreased with Pe , as shown by Fig. 3.

The above results for $\bar{\theta}$ and \bar{V} are applicable for both cases of thermal boundary conditions (prescribed temperature or prescribed heat flux at the hot wall) when the same temperature difference ΔT is maintained between the vertical walls. The temperature difference for the latter case can be obtained from equation (1) in terms of q_0 as

$$T_1 - T_2 = \left(\frac{1 - e^{-Pe}}{Pe} \right) \left(\frac{q_0 d}{k} \right). \quad (3)$$

2.2. Stability equations

If the base flow is perturbed by small disturbances, the linearized equations governing the behaviour of the perturbation quantities take the dimensionless form:

$$\frac{\partial U'}{\partial X} + \frac{\partial V'}{\partial Y} + \frac{\partial W'}{\partial Z} = 0 \quad (4)$$

$$\frac{\partial U'}{\partial \tau} - Re \frac{\partial U'}{\partial X} + Gr \bar{V} \frac{\partial U'}{\partial Y} = -\frac{\partial P'}{\partial X} + \nabla^2 U' \quad (5)$$

$$\begin{aligned} \frac{\partial V'}{\partial \tau} - Re \frac{\partial V'}{\partial X} + Gr U' \frac{\partial \bar{V}}{\partial X} + Gr \bar{V} \frac{\partial V'}{\partial Y} \\ = -\frac{\partial P'}{\partial Y} + \nabla^2 V' + \theta' \end{aligned} \quad (6)$$

$$\frac{\partial W'}{\partial \tau} - Re \frac{\partial W'}{\partial X} + Gr \bar{V} \frac{\partial W'}{\partial Y} = -\frac{\partial P'}{\partial Z} + \nabla^2 W' \quad (7)$$

$$\frac{\partial \theta'}{\partial \tau} - Re \frac{\partial \theta'}{\partial X} + Gr \bar{V} \frac{\partial \theta'}{\partial Y} + Gr V' \frac{\partial \bar{\theta}}{\partial X} = \frac{1}{Pr} \nabla^2 \theta' \quad (8)$$

subjected to the following boundary conditions:

$$\left. \begin{aligned} U' = V' = W' = 0 \quad \text{at } X = 0, 1 \\ -\frac{d\theta'}{dX} + H\theta' = 0 \quad \text{at } X = 0 \\ \theta' = 0 \quad \text{at } X = 1 \end{aligned} \right\} \quad (9)$$

where $H = \infty$ or 0 for prescribed temperature or prescribed heat flux respectively, and θ' and $\bar{\theta}$ denote the perturbed and mean quantities respectively.

By applying Squares' theorem to the present problem it can be proved that the two-dimensional disturbances are more critical than the three-dimensional ones. Therefore, only two-dimensional disturbances in the XY -plane will be considered in this analysis. Defining a stream function such that

$$U' = \frac{\partial \psi'}{\partial Y}, \quad V' = -\frac{\partial \psi'}{\partial X}. \quad (10)$$

The general solution of the stability equations can be taken in the form:

$$[\psi', \theta'] = [\psi(\tau, X), \theta(\tau, X)] e^{iaY} \quad (11)$$

where a is the wave number of the disturbances. Substituting equations (10) and (11) into equations (4)–(9) and eliminating the pressure, the following eigenvalue problem is obtained:

$$\left[\frac{\partial}{\partial \tau} - (D^2 - a^2) \right] (D^2 - a^2) \psi + iaGr \times [\bar{V}(D^2 - a^2) - D^2 \bar{V}] \psi - Re D(D^2 - a^2) \psi = D\theta \quad (12)$$

$$\left[\frac{\partial}{\partial \tau} - \frac{1}{Pr} (D^2 - a^2) \right] \theta + iaGr [\bar{V}\theta - D\bar{\theta}\psi] - Re D\theta = 0 \quad (13)$$

$$\left. \begin{aligned} \psi = D\psi = 0 \\ -D\theta + H\theta = 0 \quad (H = 0 \text{ or } \infty) \\ \theta = 0 \end{aligned} \right\} \quad \text{at } X = 0, 1 \quad (14)$$

where $D \equiv \partial/\partial X$.

The above stability problem is solved by using Galerkin's method [10]. The functions ψ and θ are represented by a series of orthogonal complex functions which satisfy the boundary conditions (14)

$$\left. \begin{aligned} \psi(\tau, X) = \sum_{i=1}^{\infty} a_i(\tau) \phi_i(X) \\ \theta(\tau, X) = \sum_{i=1}^{\infty} b_i(\tau) \theta_i(X) \end{aligned} \right\} \quad (15)$$

where the coefficients a_i and b_i are, in general, complex and unknown functions of time. The orthogonal functions $\psi_i(X)$ and $\theta_i(X)$ are chosen as in [11]:

$$\psi_i(X) = \frac{\cosh aX - \cos \zeta_i X}{\cosh a - \cos \zeta_i} - \frac{\sinh aX - \frac{a}{\zeta_i} \sin \zeta_i X}{\sinh a - \frac{a}{\zeta_i} \sin \zeta_i} \quad (16)$$

Table 1. Convergence of the results

Pr	Re	Critical parameters	Number of terms (N)				
			8	10	12	14	16
0.72	10	Gr _c	42,973	43,475	43,489	43,489	
		a _c	2.06	1.99	1.99	1.99	
	20	Gr _c	204,151	226,528	228,034	228,797	228,910
		a _c	3.32	3.17	3.16	3.16	3.16
	30	Gr _c	420,175	561,998	690,172	734,651	753,788
		a _c	5.72	5.07	4.48	4.64	5.21
5.0	1.5	Gr _c	8072	8081	8086	8086	
		a _c	1.87	1.85	1.85	1.85	
	3	Gr _c	29,278	30,743	30,772	30,784	30,784
		a _c	2.75	2.49	2.47	2.47	2.47
	4	Gr _c	46,452	62,713	65,866	66,005	66,007
		a _c	4.68	3.61	2.95	2.87	2.87
100	0.2	Gr _c	3743	4036	4105	4105	
		a _c	3.94	3.91	3.90	3.90	

$$\left. \begin{aligned} \theta_i(X) &= \sin(n\pi)X, \quad \text{for } H = \infty \\ \theta_i(X) &= \sin\left(\frac{2n-1}{2}\pi\right)X, \quad \text{for } H = 0 \end{aligned} \right\} n = 1, 2, 3, \dots$$

(17)

where ζ_i s are the positive roots of

$$[\cosh a \cos(\zeta - 1)] - \frac{a^2 - \zeta^2}{2a\zeta} \sinh a \sin \zeta = 0.$$

(18)

The above solution for ψ and θ are introduced into equations (12) and (13) and the orthogonality conditions are utilized to obtain the following matrix equation

$$\mathbf{A} \frac{d\mathbf{X}}{d\tau} + \mathbf{B}\mathbf{X} = 0$$

(19)

where $\mathbf{X} = \{a_i, b_i\}^T$ is the transpose of the coefficient vector associated with N -term expansion, and \mathbf{A} and \mathbf{B} are matrices $2N \times 2N$ with complex elements resulting from orthogonalization. The matrix eigenvalue problem (19) was solved by the complex Q - R algorithm.

The stability criteria of this system is established by determining its eigenvalues $C_n = C_r + iC_i$ (where C_i = wave speed, $n = 1, 2, \dots, 2N$). Therefore, for certain parameters of the system, Pr and Re , there is a minimum value for Gr with respect to the wave number, a . The numerical results showed that 14 terms were generally sufficient for good convergence. However, for large values of a_c Gr_c , 16 terms were used and the

convergence deteriorated as indicated in Table 1, which agrees with the previous convergence criteria [8].

3. STABILITY CRITERIA

Before discussing the stability characteristics of the combined forced cross and natural flow inside a vertical slot, it is interesting to compare our results, for $Re \rightarrow 0$ with previous natural convection solutions. Table 2 presents a comparison between published critical conditions and the present investigation numerical values, at various Prandtl numbers. It can be seen that there is a very good agreement between the results at the same conditions which provide a further check on the numerical accuracy.

The neutral states of the present stability problem for $Pr = 0.72$ and 5 are illustrated in Figs 4 and 5 respectively. For isothermal boundary conditions, $Pr = 0.72$ and $Re \leq 2$, the neutral stability curves are characterized by one minimum referring to the hydrodynamic mode of instability. As Re increases from 2 to 5 two minima were obtained in the neutral curves, one corresponding to the hydrodynamic mode which is characterized by the higher wave number, and the other corresponding to the thermal mode of relatively lower wave number. In this range of Re , the critical mode is the hydrodynamic one; however, for $Re = 6$ the opposite is true. For higher values of Re , i.e. $Re \geq 8$, it seems that the instability is characterized by the thermal mode only. On the other hand, for $Pr = 5$, the transition from the hydrodynamic, as a critical

Table 2. Comparison between present investigation with natural convection results

References	Lower limit of Pr			Upper limit of Pr		
	Re	Pr	Gr	Re	Pr	Gr
Present investigation	0.01	0.72	8011	0.001	100	737
[9]	0	0.72	8000		—	—
[12]	0	0.71	8027		—	—
[13]	0	0.71	8038	0	100	750

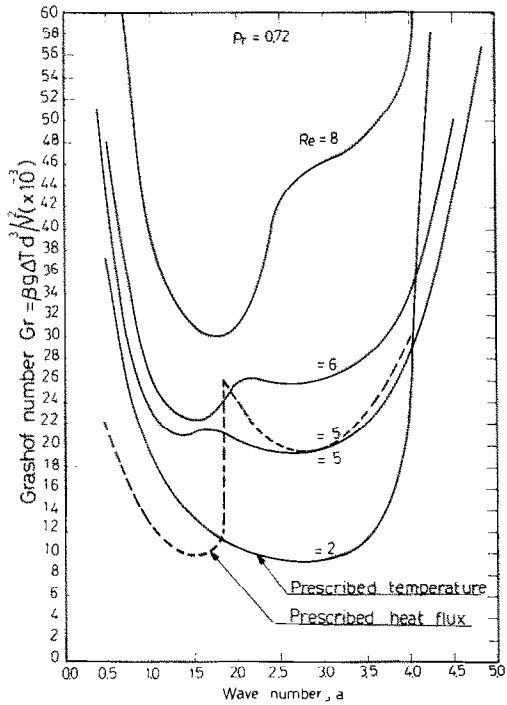


FIG. 4. Neutral stability curves for $Pr = 0.72$ at various Re values.

mode, to the thermal one takes place at a much lower value of Re . Lastly, as Pr increases to 100, the thermal mode becomes the only critical one over the entire range of Re .

For prescribed heat flux boundary condition, few samples of neutral states are presented to illustrate the difference in the critical condition, as well as the critical mode between this case and the isothermal one. For $Pr = 0.72$, the transition occurs at $Re < 4$ corresponding to $Re = 6$ in the isothermal case. Furthermore, for $Pr = 5$, a more significant influence of the bound-

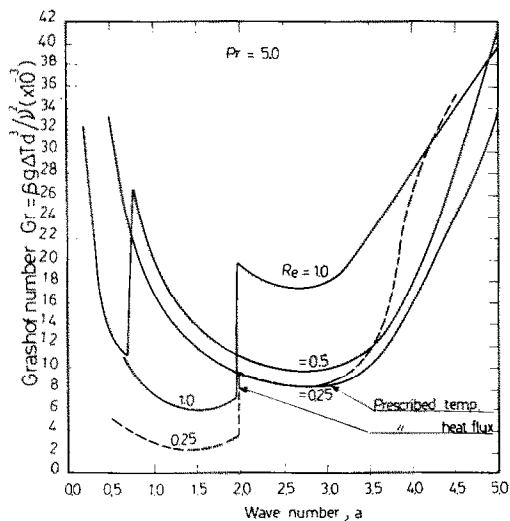


FIG. 5. Neutral stability curves for $Pr = 5.0$ at various Re values.

ary condition is observed where the thermal mode is more critical for all investigated values of Re .

Therefore, one of the interesting findings of this study is that the transition from the hydrodynamic to the thermal mode occurs at smaller Prandtl numbers than previously reported [8]. In addition, these neutral stability curves are analogous to those for boundary layer regime [3, 11]. This phenomenon is not surprising when the temperature profiles of the present problem are analysed.

The variations of the critical Grashof number, Gr_c , with the Peclet number, Pe , for the considered values of the Prandtl number are shown in Fig. (6). The results indicate that, for $Pe < 4$, the flow is rapidly stabilized against the hydrodynamic mode and destabilized against thermal mode. In general, there are two opposing contributions of the cross flow on the stability problem. On one hand, a stabilizing effect due to the damping inertia force resulting from the base flow velocity. On the other hand, a destabilizing effect due to the increasing buoyancy force resulting from the steep density gradient. Depending on the order of magnitude of each contribution, which is a function of the fluid properties and the thermal boundary conditions, the flow will be either stabilized or destabilized. On the limiting case for very small Prandtl numbers, where the disturbances receive energy from the base flow velocity, the stabilizing contribution is most significant. Conversely, for very large Prandtl numbers, where more energy feeds the disturbances through the buoyancy field, the destabilizing effect increases. It is to be noted that for $Pr = 5$, the steep destabilizing effect is due to the transition from the hydrodynamic to the

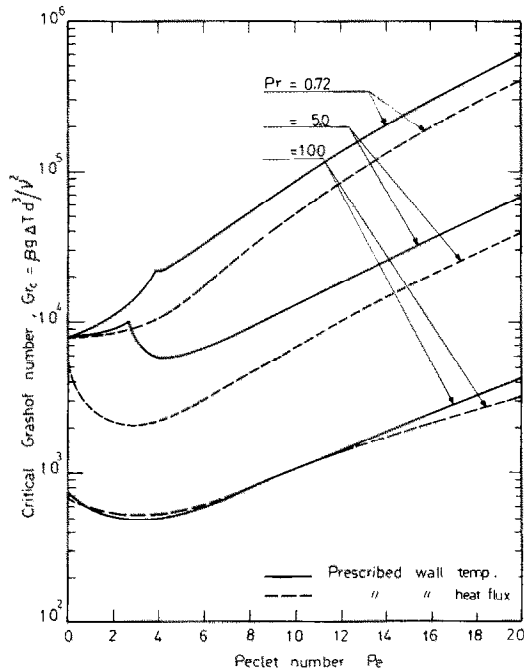


FIG. 6. Effect of cross flow on the critical Grashof number for fluids in a vertical slot, $Pr = 0.72, 5$ and 100 .

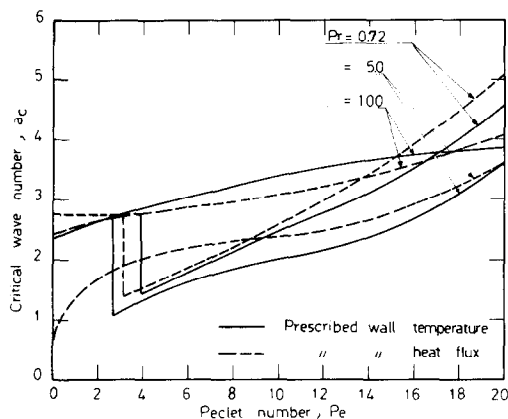


FIG. 7. Variation of the critical wave number with cross flow for $Pr = 0.72, 5, 100$.

thermal mode as previously indicated. Lastly, as the Peclet number increases, i.e. $Pe > 4$, it appears that the stabilizing effect is more pronounced than the destabilizing contribution.

The effect of the cross flow on the characteristics of the secondary flow at the neutral states, as expressed by the critical wave number, a_c , and the wave speed, C_i , is shown in Figs. (7) and (8). For the hydrodynamic mode, the wavelength ($L = 2\pi/a_c$) is nearly independent on Re and equal to 9.3 times the gap thickness which agrees with the previous work [8] at the limiting case of $Pe = 0$. In addition, the critical wave speed is no longer stationary but has a very small velocity (compared to that of the base flow) which increases with the cross flow. On the other hand, as the transition to the thermal

mode occurs both the wave length and the wave speed suddenly increase and then decrease as Pe increases. It is to be noted that the effect of buoyancy in this problem also increases the wave speed compared to the maximum base flow velocity such that for $Pr = 100$ this difference is significant.

4. CONCLUSIONS

(1) The onset of instability of the combined forced cross flow and the natural flow inside a vertical slot is greatly dependent on Pr , Re and the thermal boundary conditions.

(2) The transition from the hydrodynamic to the thermal mode occurs at $Pr < 12.7$, depending on Re .

(3) The effect of the cross flow can be classified in two regions. For $Pe < 4$, the flow is stabilized against the hydrodynamic mode and destabilized against the thermal mode. Conversely, for $Pe > 4$.

(4) The flow is generally more stabilized for the isothermal walls than for the prescribed heat flux case. In addition, the transition criteria occurs at lower Re for the latter case.

(5) The wave speed of the hydrodynamic mode is characterized by a very small value which increases with cross flow. However, in the thermal mode it is nearly of the same order of magnitude as that of the base flow, but decreases with cross flow.

(6) The wave size of the hydrodynamic mode is nearly independent on the cross flow. Conversely, it decreases with cross flow in the thermal mode.

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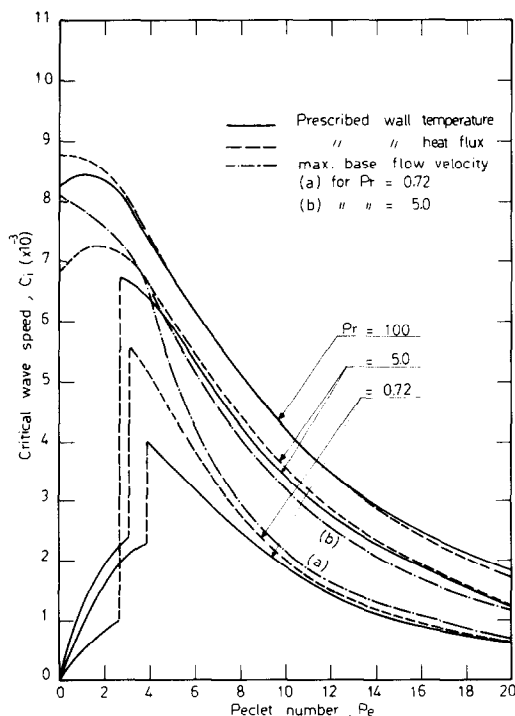


FIG. 8. Variation of the maximum base flow velocity and the wave speed with cross flow.

STABILITE D'UN ECOULEMENT TRANSVERSAL MIXTE DANS UNE FENTE VERTICALE

Résumé—Cette étude concerne la stabilité de l'écoulement mixte de fluide contenu dans une fente étroite verticale avec parois poreuses minces. Les paramètres couvrent des domaines étendus en nombres de Prandtl et de Péclet, soit $0,72 \leq Pr \leq 100$ et $0 \leq Pe \leq 20$. De plus ce problème est analysé pour des conditions aux limites thermiques différentes sur la paroi chaude, c'est-à-dire température ou flux thermiques imposé. La comparaison de cette étude avec le problème de convection naturelle montre que pour $Pr \geq 0,72$ et $Pe \leq 4$, l'écoulement transversal a un effet destabilisateur, tandis que pour $Pe \geq 4$ l'écoulement est stable. L'accroissement de l'écoulement transversal déplace le mode d'instabilité pour des fluides ayant un faible nombre de Prandtl depuis la stationarité vers les ondes mobiles.

STABILITÄT EINER GEMISCHTEN (NATÜRLICHEN UND ERZWUNGENEN) QUERSTRÖMUNG IN EINEM SENKRECHTEN SPALT

Zusammenfassung—Diese Untersuchung behandelt die Stabilität einer gemischten (natürlichen und erzwungenen) Querströmung von Fluiden in einem senkrechten, schmalen Spalt mit dünnen, porösen Wänden. Die Prandtl- und Peclet-Zahlen erstrecken sich über weite Bereiche: $0,72 \leq Pr \leq 100$ bzw. $0 \leq Pe \leq 20$. Darüberhinaus wird dieses Problem für zwei unterschiedliche thermische Randbedingungen an der heißen Wand untersucht, nämlich aufgeprägte Temperatur und Wärmestromdichte. Beim Vergleich der Ergebnisse dieser Untersuchung mit denjenigen des Problems der natürlichen Konvektion wurde herausgefunden, daß für $Pr > 0,72$ und $Pe < 4$ die Querströmung einen destabilisierenden Einfluß hat, währenddessen die Strömung für $Pe > 4$ stabilisiert wird. Außerdem verschiebt ein Zunehmen der Querströmung für Fluide mit kleinen Prandtl-Zahlen die Art der Instabilität von stationären zu wandernden Wellen.

УСТОЙЧИВОСТЬ СМЕШАННЫХ ЕСТЕСТВЕННОГО И ВЫНУЖДЕННОГО ПОПЕРЕЧНЫХ ТЕЧЕНИЙ В ВЕРТИКАЛЬНОЙ ШЕЛИ

Аннотация—Изучается устойчивость смешанного естественного и вынужденного поперечных течений жидкости в вертикальной узкой щели с тонкими пористыми стенками. Параметры системы охватывают широкий диапазон чисел Прандтля и Пекле: $0,72 \leq Pr \leq 100$ и $0 \leq Pe \leq 20$, соответственно. Эта задача рассматривается также для двух различных тепловых граничных условий у горячей стенки, т.е. от заданных температуры и теплового потока. При сравнении результатов исследования с задачей для естественной конвекции найдено, что для $Pr > 0,72$ и $Pe < 4$ поперечное течение оказывает дестабилизирующее действие, в то время как при $Pe > 4$ течение стабилизируется. Более того, для жидкостей, имеющих малые числа Прандтля, режим неустойчивости сдвигается от стационарного к режиму бегущих волн.